Properties of spin-triplet, even-parity superconductors

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The physical consequences of the spin-triplet, even-parity pairing that has been predicted to exist in disordered two-dimensional electron systems are considered in detail. We show that the presence of an attractive interaction in the particle-particle spin-triplet channel leads to an instability of the normal metal that competes with the localizing effects of the disorder. The instability is characterized by a diverging length scale, and has all of the characteristics of a continuous phase transition. The transition and the properties of the ordered phase are studied in mean-field theory, and by taking into account Gaussian fluctuations. We find that the ordered phase is indeed a superconductor with an ordinary Meissner effect and a free energy that is lower than that of the normal metal. Various technical points that have given rise to confusion in connection with this and other manifestations of odd-gap superconductivity are also discussed.

I. INTRODUCTION

The possibility of superconductivity with a gap function that is an odd function of time or frequency has been the subject of some attention lately. The concept was introduced by Berezinskii[1] in the context of ^{3}He, and it was more recently revived in connection with two-dimensional (2-D) electron fluids in semiconductors,[2] and with high-T_c superconductors.[3] In Berezinskii's original work, as well as in Refs. 2,4, the case of spin-triplet, even-parity pairing was considered, while Refs. 3 also discussed the case of spin-singlet, odd-parity pairing. In either case, the gap function being an odd function of the frequency ensures that the Pauli principle is obeyed.

In these references, some properties of odd-gap superconductors have been explored, but no complete analysis of the phase transition, or of the superconducting phase, has been given. Furthermore, several fundamental questions concerning the stability of such a superconducting phase, and whether an odd-gap superconductor can indeed be superconducting, have led to considerable confusion.

In the present paper we discuss these issues. For definiteness, we will analyze the mechanism for spin-triplet, even-parity superconductivity discovered in Refs. 4, but many of our conclusions apply to other odd-gap superconductors as well. The paper is organized as follows. In Sec. II we give a field-theoretic formulation of the problem and discuss the stability properties of the field theory. In Sec. III we show that in the presence of an attractive interaction in the particle-particle spin-triplet channel, there is an instability of the normal conducting phase. This instability has all of the properties of a second order phase transition, with a diverging length scale, a diverging order parameter susceptibility, etc. In Sec. IV we develop and analyze a mean-field theory of the transition. In particular, we determine the critical behavior on either side of the transition. We then determine the properties of the ordered phase in a Gaussian approximation. We show that the ordered phase is a superconductor, with a Meissner effect, and a real part of the conductivity that has a delta-function contribution, just as in BCS superconductors. In Sec. V we conclude with a summary and a general discussion of our results.

II. FIELD-THEORETIC FORMULATION OF THE PROBLEM

A. Q-matrix theory for fermions

Our starting point is a general field theory for electrons. For any fermionic system, the partition function can be written[5]

$$Z = \int D[\bar{\psi},\psi]\ \exp\left(S[\bar{\psi},\psi]\right) \quad , \tag{2.1a}$$

where S is the action in terms of the fermionic (i.e., Grassmann valued) fields $\bar{\psi}$ and ψ. We consider an action that consists of a free-fermion part S_0, a part $S_{\rm dis}$ describing the interaction of the electrons with quenched disorder, and a part $S_{\rm int}$ describing the electron-electron interaction,

$$S = S_0 + S_{\rm dis} + S_{\rm int} \quad , \tag{2.1b}$$

Each field ψ or $\bar{\psi}$ carries a Matsubara frequency index n, a spin index $\sigma =\uparrow,\downarrow$, and, if the quenched disorder is dealt with by means of the replica trick, a replica index α. For our purposes it is useful to introduce a matrix of bilinear products of the fermion fields,

$$B_{12} = \frac{i}{2}\begin{pmatrix} -\psi_{1\uparrow}\bar{\psi}_{2\uparrow} & -\psi_{1\uparrow}\bar{\psi}_{2\downarrow} & -\psi_{1\uparrow}\psi_{2\downarrow} & \psi_{1\uparrow}\psi_{2\uparrow} \\ -\psi_{1\downarrow}\bar{\psi}_{2\uparrow} & -\psi_{1\downarrow}\bar{\psi}_{2\downarrow} & -\psi_{1\downarrow}\psi_{2\downarrow} & \psi_{1\downarrow}\psi_{2\uparrow} \\ \bar{\psi}_{1\downarrow}\bar{\psi}_{2\uparrow} & \bar{\psi}_{1\downarrow}\bar{\psi}_{2\downarrow} & \bar{\psi}_{1\downarrow}\psi_{2\downarrow} & -\bar{\psi}_{1\downarrow}\psi_{2\uparrow} \\ -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\uparrow} & -\bar{\psi}_{1\uparrow}\bar{\psi}_{2\downarrow} & -\bar{\psi}_{1\uparrow}\psi_{2\downarrow} & \bar{\psi}_{1\uparrow}\psi_{2\uparrow} \end{pmatrix} \cong Q_{12} \quad . \tag{2.2}$$

where all fields are understood to be taken at position $\mathbf{x}$, and $1 \equiv (n_1, \alpha_1)$, etc. The matrix elements of B commute with one another, and are therefore isomorphic to classical or number-valued fields that we denote by Q.[6] This isomorphism maps the adjoint operation on products of fermion fields, which is denoted above by an overbar, on the complex conjugation of the classical fields. We use the isomorphism to constrain B to the classical field Q, and exactly rewrite the partition function[7]

$$\begin{aligned} Z &= \int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]} \int D[Q]\,\delta[Q-B] \\ &= \int D[\bar{\psi},\psi]\; e^{S[\bar{\psi},\psi]} \int D[Q]\,D[\widetilde{\Lambda}]\; e^{\mathrm{Tr}\,[\widetilde{\Lambda}(Q-B)]} \\ &\equiv \int D[Q]\,D[\widetilde{\Lambda}]\; e^{\mathcal{A}[Q,\widetilde{\Lambda}]} \quad . \end{aligned} \tag{2.3}$$

$\widetilde{\Lambda}$ is an auxiliary bosonic matrix field that serves to enforce the functional delta-constraint in the first line of Eq. (2.3), and the last line defines the action $\mathcal{A}$. The matrix elements of both Q and $\widetilde{\Lambda}$ are spin-quaternions (i.e., elements of $\mathcal{Q}\times\mathcal{Q}$ with $\mathcal{Q}$ the quaternion field). From Eq. (2.2) we see that expectation values of the Q matrix elements yield local Green functions, and Q-Q correlation functions describe four-fermion correlation functions. The physical meaning of $\widetilde{\Lambda}$ is that its expectation value plays the role of a self energy (see Ref. 7 and Sec. IV below).

It is convenient to expand the 4×4 matrix in Eq. (2.2) in a spin-quaternion basis,

$$Q_{12}(\mathbf{x}) = \sum_{r,i=0,3} (\tau_r \otimes s_i)\, {}^{i}_{r}Q_{12}(\mathbf{x}) \tag{2.4}$$

and analogously for $\widetilde{\Lambda}$. Here $\tau_0 = s_0 = \mathbb{1}_2$ is the 2×2 unit matrix, and $\tau_j = -s_j = -i\sigma_j$, $(j=1,2,3)$, with $\sigma_{1,2,3}$ the Pauli matrices. In this basis, $i=0$ and $i=1,2,3$ describe the spin singlet and the spin triplet, respectively. An explicit calculation reveals that $r=0,3$ corresponds to the particle-hole channel (i.e., products $\bar{\psi}\psi$), while $r=1,2$ describes the particle-particle channel (i.e., products $\bar{\psi}\bar{\psi}$ or $\psi\psi$). We will be particularly interested in the matrix elements of ${}^{1}_{1}Q$, for which the isomorphism expressed in Eq. (2.2) reads[8]

$$\begin{aligned} {}^{1}_{1}Q_{12}(\mathbf{x}) \cong \frac{i}{8}\,[&\psi_{1\uparrow}(\mathbf{x})\psi_{2\uparrow}(\mathbf{x}) - \psi_{1\downarrow}(\mathbf{x})\psi_{2\downarrow}(\mathbf{x}) \\ &+ \bar{\psi}_{1\downarrow}(\mathbf{x})\bar{\psi}_{2\downarrow}(\mathbf{x}) - \bar{\psi}_{1\uparrow}(\mathbf{x})\bar{\psi}_{2\uparrow}(\mathbf{x})] \quad . \end{aligned} \tag{2.5}$$

From the structure of Eq. (2.2) one obtains the following formal symmetry properties of the Q matrices,[7]

$$ {}^{0}_{r}Q_{12} = (-)^r\, {}^{0}_{r}Q_{21} \quad , \quad (r=0,3) \quad , \tag{2.6a}$$
$$ {}^{i}_{r}Q_{12} = (-)^{r+1}\, {}^{i}_{r}Q_{21} \;,\; (r=0,3;\; i=1,2,3) \quad , \tag{2.6b}$$
$$ {}^{0}_{r}Q_{12} = {}^{0}_{r}Q_{21} \quad , \quad (r=1,2) \quad , \tag{2.6c}$$
$$ {}^{i}_{r}Q_{12} = -{}^{i}_{r}Q_{21} \quad , \quad (r=1,2;\; i=1,2,3) \quad , \tag{2.6d}$$
$$ {}^{i}_{r}Q^{*}_{12} = -{}^{i}_{r}Q^{\alpha_1\alpha_2}_{-n_1-1,-n_2-1} \quad . \tag{2.6e}$$

Here the star in Eq. (2.6e) denotes complex conjugation.

By using the delta constraint in Eq. (2.3) to rewrite all terms that are quartic in the fermion field in terms of Q, we can achieve an integrand that is bilinear in ψ and $\bar{\psi}$. The Grassmannian integral can then be performed exactly, and we obtain for the action $\mathcal{A}$

$$\begin{aligned} \mathcal{A}[Q,\widetilde{\Lambda}] = \mathcal{A}_{\mathrm{dis}} + \mathcal{A}_{\mathrm{int}} &+ \frac{1}{2}\,\mathrm{Tr}\,\ln\left(G_0^{-1} - i\widetilde{\Lambda}\right) \\ &+ \int d\mathbf{x}\;\mathrm{tr}\,\left(\widetilde{\Lambda}(\mathbf{x})\,Q(\mathbf{x})\right) \quad . \end{aligned} \tag{2.7a}$$

Here

$$G_0^{-1} = -\partial_\tau + \partial_{\mathbf{x}}^2/2m + \mu \quad , \tag{2.7b}$$

is the inverse free electron Green operator, with ∂_τ and $\partial_{\mathbf{x}}$ derivatives with respect to imaginary time and position, respectively, m is the electron mass, and μ is the chemical potential. Tr denotes a trace over all degrees of freedom, including the continuous position variable, while tr is a trace over all those discrete indices that are not explicitly shown. The electron-electron interaction $\mathcal{A}_{\mathrm{int}}$ is conveniently decomposed into four pieces that describe the interaction in the particle-hole and particle-particle spin-singlet and spin-triplet channels.[7] For the purposes of the present paper, we need only the particle-particle spin-triplet channel interaction explicitly. In Ref. 4 it was shown that in any quenched disordered, interacting electron system, there is an attractive interaction in the particle-particle spin-triplet channel of the form

$$\begin{aligned} \mathcal{A}_{\mathrm{int}}^{\mathrm{p-p,t}} = -\pi N_{\mathrm{F}} \int d\mathbf{x}\; T \sum_{n_1,n_2,n_3,n_4} \delta_{n_1+n_2,n_3+n_4} \\ \times \widetilde{K}_{n_1,n_2;n_3,n_4} \sum_{r=1,2}\sum_{i=1}^{3}\sum_{\alpha} {}^{i}_{r}Q^{\alpha\alpha}_{n_1n_2}\, {}^{i}_{r}Q^{\alpha\alpha}_{n_3n_4} \quad . \end{aligned} \tag{2.8a}$$

In $D=2$, the effective interaction potential is

$$\begin{aligned} \widetilde{K}_{n_1,n_2;n_3,n_4} = \frac{1}{4}\,(&K_{n_1,n_2;n_3,n_4} - K_{n_2,n_1;n_3,n_4} \\ &- K_{n_1,n_2;n_4,n_3} + K_{n_2,n_1;n_4,n_3}) \quad , \end{aligned} \tag{2.8b}$$

where[9]

$$K_{n_1,n_2;n_3,n_4} = y \ln \left| \frac{n_1 - n_3}{n_2 - n_3} \right| \quad , \tag{2.8c}$$

with a positive coupling constant $y > 0$ that depends both on the disorder strength and on the coupling constants in the particle-hole interaction channels. Although both the disorder and the particle-hole channel interactions are necessary to produce the particle-particle spin-triplet interaction, none of the points to be investigated in this paper qualitatively depends on either one of them other than through the existence of $\widetilde{K}$. For the Gaussian theory that we will consider, the only other effect of the disorder that is relevant to our dicussion is that it replaces some free-electron correlation functions by diffusive ones, which changes the exponents in certain scaling relations. For notational simplicity, and in order to keep our discussion technically as simple as possible, in what follows we therefore neglect all contributions to the action that are nonessential for our purposes. We thus work with a system given by Eq. (2.7a) with $\mathcal{A}_{\rm dis} = 0$ and $\mathcal{A}_{\rm int} = \mathcal{A}_{\rm int}^{p-p,t}$ and drop the replica indices on all fields. In cases where diffusive correlations make a difference, we will mention this explicitly and restore diffusive scaling. Our restriction to a Gaussian approximation purposely neglects the localizing effects of the disorder. We will come back to this point in Sec. V C.

Note that Eq. (2.8c) means that in time space, the interaction between superconducting fluctuations is long-ranged. Consequently, the critical behavior, and in particular the critical exponents discussed in Secs. III and IV A below, depend explicity on the detailed form of the kernel. However, our qualitative results, in particular the spontaneous symmetry breaking and the existence of an ordinary quantum critical point that marks the onset of superconducting long-range order, we expect to be generic. We also note that in Eqs. (2.8) above, as in Ref. 4, we have neglected any wavenumber dependences of the effective interaction potential. Any nontrivial, i.e. nonanalytic, wavenumber dependence would correspond to an interaction that is long-ranged in real space. Such long-ranged interactions are known to stabilize mean-field critical behavior,[10] and will thus increase the range of validity of our mean-field theory. We therefore do not expect our neglecting the wavenumber dependence of K to qualitatively affect our results.

B. The Fermi-liquid saddle point, and Gaussian fluctuations

In Ref. 7 it was shown that the above Q-$\widetilde{\Lambda}$ field theory possesses a saddle-point solution that describes a free Fermi gas (or a disordered Fermi liquid if we had not dropped the disorder and particle-hole channel interaction contributions to the action). That is, the saddle-point equations

$$\left.\frac{\delta \mathcal{A}}{\delta Q}\right|_{Q_{\rm sp},\widetilde{\Lambda}_{\rm sp}} = \left.\frac{\delta \mathcal{A}}{\delta \widetilde{\Lambda}}\right|_{Q_{\rm sp},\widetilde{\Lambda}_{\rm sp}} = 0 \quad , \tag{2.9}$$

are solved by the *ansatz*

$$\left.{}^{i}_{r}Q_{12}(\mathbf{x})\right|_{\rm sp} = \delta_{12}\,\delta_{r0}\,\delta_{i0}\,Q_{n_1} \quad , \tag{2.10a}$$

$$\left.{}^{i}_{r}\widetilde{\Lambda}_{12}(\mathbf{x})\right|_{\rm sp} = \delta_{12}\,\delta_{r0}\,\delta_{i0}\,\Lambda_{n_1} \quad , \tag{2.10b}$$

with

$$Q_n = \frac{i}{2V} \sum_{\mathbf{p}} G_n^0(\mathbf{p}) \quad , \tag{2.11a}$$

$$\Lambda_n = 0 \quad . \tag{2.11b}$$

The saddle-point Green function G_n^0 is simply the one for free electrons,

$$G_n^0(\mathbf{p}) = (i\omega_n - \xi_{\mathbf{p}})^{-1} \quad , \tag{2.12}$$

with $\xi_{\mathbf{p}} = \mathbf{p}^2/2m - \mu$.

We next consider the Gaussian fluctuations about this saddle point. Proceeding as in Ref. 7, we write

$$Q = Q_{\rm sp} + \delta Q \quad , \tag{2.13a}$$

$$\tilde{\Lambda} = \tilde{\Lambda}_{\rm sp} + \delta\tilde{\Lambda} \quad , \tag{2.13b}$$

and introduce a new field $\bar{\Lambda}$ by

$$\bar{\Lambda}_{12} = \frac{1}{2}\,\varphi_{12}\,\widetilde{\Lambda}_{12} + Q_{12} \quad , \tag{2.14a}$$

with

$$\varphi_{nm}(\mathbf{k}) = \frac{1}{V} \sum_{\mathbf{p}} G_{\rm sp}(\mathbf{p},\omega_n)\, G_{\rm sp}(\mathbf{p}+\mathbf{k},\omega_m) \quad , \tag{2.14b}$$

a generalized Lindhard function. $\bar{\Lambda}$ has been chosen so that Q and $\bar{\Lambda}$ decouple in the Gaussian action. Expanding to second order in the fluctuating fields δQ and $\delta\bar{\Lambda}$, we obtain the Gaussian action

$$\mathcal{A}_{\rm G} = \frac{4}{V} \sum_{\mathbf{k}} \sum_{1,2} \sum_{r,i} \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \varphi_{12}^{-1}(\mathbf{k}) \left[{}^{i}_{r}(\delta\bar{\Lambda})_{12}(\mathbf{k}) \times {}^{i}_{r}(\delta\bar{\Lambda})_{12}(-\mathbf{k}) - {}^{i}_{r}(\delta Q)_{12}(\mathbf{k})\, {}^{i}_{r}(\delta Q)_{12}(-\mathbf{k}) \right] + \mathcal{A}_{\rm int}^{\rm p-p,t}[\delta Q] \quad , \tag{2.15}$$

where the symbol $\begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r$ is equal to $+1$ for $r = 0, 3$, and -1 for $r = 1, 2$.

Keeping the disorder part of the action results in the inverse Lindhard function $\varphi_{12}^{-1}(\mathbf{k})$ in Eq. (2.15) being replaced by[7]

$$\mathcal{D}_{12}^{-1}(\mathbf{k}) = \varphi_{12}^{-1}(\mathbf{k}) - 1/\pi\, N_{\rm F}\, \tau \quad , \tag{2.16a}$$

with τ the elastic scattering mean-free time. For small wavenumbers and small $\omega_{n_1} - \omega_{n_2}$ with $n_1 n_2 < 0$, $\mathcal{D}$ is diffusive,

$$\mathcal{D}_{12}(\mathbf{k}) = \frac{\pi N_{\rm F}}{D\,\mathbf{k}^2 + |\omega_{n_1} - \omega_{n_2}|} \quad , \tag{2.16b}$$

with D the Boltzmann value of the diffusion constant. Whenever it makes a difference in the results, we will use the diffusive propagator $\mathcal{D}$ instead of the free fermion propagator φ.

In order to completely define the field theory, one also must specify the integration contours for the functional integral, Eq. (2.3). The most obvious choice would be to integrate over the space of all matrices that obey Eqs. (2.6). However, this creates the following problem. According to Eqs. (2.6d), (2.6e), ${}^{1}_{1}Q_{n-1,-n}$ is real, and so are all other particle-particle spin-triplet components of $Q_{n-1,-n}$. However, according to Eq. (2.15), this direction in the complex Q-plane is unstable, and the direction of steepest descent is along the imaginary axis.[11] In order to do the integral by the saddle-point method, the contour for the integration over the ${}^{1,2,3}_{1,2}Q$ must therefore be deformed so that it passes through the saddle point in the direction of the imaginary axis. At least within perturbation theory, this just amounts to formally doing the Gaussian integral without worrying about convergence problems. A necessary condition for this procedure to be valid is that the resulting theory reproduces the perturbative results obtained within the underlying fermionic theory. An easy and convenient check is provided by the so-called weak-localization correction to the conductivity of non-interacting electrons. For our purposes, a simple structural check suffices and is provided in Appendix A. The check is affirmative, and suggests that we can safely ignore convergence questions relating to the Gaussian integrals that occur in our field theory. A related point will become important in Sec. IV below.

III. INSTABILITY OF THE NORMAL METAL

A. The Gaussian eigenvalue problem in the normal phase

We now turn to the full Gaussian action in an expansion about the Fermi liquid saddle-point, Eq. (2.15). Integrating out the auxiliary field $\bar{\Lambda}$ just contributes a multiplicative constant to the partition function, and we are left with a quadratic form in δQ. Let us consider the particle-particle spin-triplet part of the action, e.g., the channel $r = i = 1$. Since the action is diagonal in all indices except the frequency, we can drop the r and i indices as well as the wavenumber dependence, and consider a quadratic form

$$\sum_{1,2,3,4} \delta Q_{12}\, M_{12,34}\, \delta Q_{34} \quad , \tag{3.1a}$$

where the matrix M has the structure

$$M_{12,34} = -a_{12}\,\delta_{13}\,\delta_{24} - b_{1-3,2-3}\,\delta_{1+2,3+4} \quad . \tag{3.1b}$$

Here b is proportional to the interaction potential K, Eq. (2.8c), and a_{12} is either φ_{12}^{-1}, Eq. (2.14b), or $\mathcal{D}_{12}^{-1}$, Eq. (2.16a), depending on whether or not we keep the disorder explicitly. It is useful to consider M as a matrix with composite indices $(1,2)$ and $(3,4)$, and to study the eigenvalue problem

$$\sum_{1',2'} M_{12,1'2'}\, f_{1'2'} = \lambda\, f_{12} \quad , \tag{3.2}$$

with eigenvalues λ and eigenfunctions f_{12}. For reasons that will soon become apparent, it is advantageous to transform to a different basis of eigenfunctions g defined by

$$g_{12} = \sum_{3,4} b_{1-3,2-3}\,\delta_{1+2,3+4}\, f_{34} \quad . \tag{3.3}$$

In terms of the g, the eigenvalue problem reads

$$g_{12} = -\sum_{1'} \frac{b_{1-1',2-1'}}{a_{1',-1'+(1+2)} + \lambda}\; g_{1',-1'+(1+2)} \quad . \tag{3.4a}$$

Now we write $\omega_{n_2} = -\omega_{n_1} + \Omega_n$, analytically continue to real frequencies at $T = 0$, and put the wavenumber dependence back in. Then we obtain the eigenvalue equation in the form

$$g_\lambda(\omega;\mathbf{p},\Omega) = -y \int_{-\infty}^{\infty} dx\; \ln\left|\frac{\omega - x}{-\omega - x + \Omega}\right| \times \frac{1}{a(x;\mathbf{p},\Omega) + \lambda}\; g_\lambda(x;\mathbf{p},\Omega) \quad . \tag{3.4b}$$

Here y is the coupling constant of Eq. (2.8c), scaled by an appropriate factor. In order to check for an instability of the metallic phase, we need to look for a zero eigenvalue. From the structure of Eq. (3.4b) it is clear that the first zero eigenvalue appears for $\mathbf{p} = \Omega = 0$. We therefore specialize to zero momentum and external frequency. We then have $a(x;\mathbf{p} = 0, \Omega = 0) \propto |x|$, irrespective of whether we use $a = \varphi^{-1}$ or $a = \mathcal{D}^{-1}$. Absorbing a constant factor into the eigenvalue λ and into the coupling constant y, we then get the eigenvalue equation in the form

$$g_\lambda(\omega) = y \int_0^\infty \frac{dx}{x + \lambda}\; \ln\left|\frac{x + \omega}{x - \omega}\right|\, g_\lambda(x) \quad , \tag{3.5}$$

with $g_\lambda(\omega) \equiv g_\lambda(\omega;\mathbf{p} = 0, \Omega = 0)$. Notice that Eq. (3.5) is a generalization of the gap equation in Ref. 4: The critical eigenfunction, $g_{\lambda=0}(\omega)$, obeys the same integral equation as the critical gap function. This is the first indication that the long-range order implied by a nonzero gap function in Ref. 4 can indeed be understood in terms of a conventional continuous phase transition that is triggered by an instability of the metallic phase.

B. Solution of the eigenvalue problem

The integral equation, Eq. (3.5), is very hard to solve analytically. We therefore make the same approximation as was done for the gap equation in Ref. 4, namely replacing the logarithmic kernel by a rational one with a similar overall behavior:

$$\ln\left|\frac{x+\omega}{x-\omega}\right| \to \frac{2x}{\omega}\,\Theta(x-\omega) + \frac{2\omega}{x}\,\Theta(\omega-x) \quad . \tag{3.6}$$

This allows to rewrite the integral equation as an ordinary differential equation,

$$\frac{d^2}{d\omega^2}\,g_\lambda(\omega) + \frac{1}{\omega}\,\frac{d}{d\omega}\,g_\lambda(\omega) + \left[\frac{4y}{\omega(\omega+\lambda)} - \frac{1}{\omega^2}\right] g_\lambda(\omega) = 0. \tag{3.7}$$

It is useful to rewrite this ODE in the form

$$\frac{d^2}{dz^2}\,w(z) + p(z)\,\frac{d}{dz}\,w(z) + q(z)\,w(z) = 0 \quad , \tag{3.8a}$$

with

$$p(z) = \frac{1}{z} \quad , \tag{3.8b}$$

$$q(z) = \frac{-1}{z^2} + \frac{t-1}{z} + \frac{1-t}{z-1} \quad , \tag{3.8c}$$

where $t = 1 - 4y$, and the solution $w(z)$ of the ODE determines the eigenfunction via

$$g_\lambda(\omega) = \lambda^2\, w(-\omega/\lambda) \quad . \tag{3.8d}$$

The ODE, Eq. (3.8a), is Fuchsian with three regular singular points at $z = 0$, $z = 1$, and $z = \infty$. It can thus be transformed into a hypergeometric equation.[12] Before we look at the general solution, let us consider the case $\lambda = 0$, where the two solutions of Eq. (3.7) are

$$g^{(1)}_{\lambda=0} = \omega^{-\sqrt{t}} \quad , \quad g^{(2)}_{\lambda=0} = \omega^{\sqrt{t}} \quad . \tag{3.9}$$

In order to obtain a solution that is well-behaved everywhere, we therefore must require $t(\lambda = 0) = 0$. More generally, the requirement of a well-behaved eigenfunction leads to t being a function of λ, as we now proceed to show. The solution of Eq. (3.8a) that is well-behaved for $\omega \to \infty$ can be written

$$g_\lambda(\omega) = \left(\frac{\omega}{\lambda}\right)^{-\sqrt{t}} \frac{1}{(1+\lambda/\omega)^{1+\sqrt{t}}} \times F\left(1+\sqrt{t}, 2+\sqrt{t}; 3; \frac{1}{1+\lambda/\omega}\right) \quad , \tag{3.10}$$

with F a hypergeometric function. For later reference we also list the large-frequency behavior of the original eigenfunction f, related to g by Eq. (3.3). By an analysis analogous to the one above for g one finds

$$f_\lambda(\omega \to \infty) \propto (\omega/\lambda)^{-1-\sqrt{t}} \quad . \tag{3.11}$$

C. The correlation length, and the exponents ν, η, and γ

Let us consider the eigenfunction, Eq. (3.10), in the limit $\lambda \to 0$. Anticipating that $t \to 0$ as well in that limit, we expand in t and find asymptotically

$$g_{\lambda\to 0}(\omega) = -(\omega/\lambda)^{-\sqrt{t}}/\sqrt{t} \quad . \tag{3.12}$$

We know, however, that in this limit the eigenfunction must merge with the solution $g^{(1)}_{\lambda=0}$, Eq. (3.9). This yields λ as a function of t for asymptotically small t,

$$\lambda(t \to 0) \propto t^{1/2\sqrt{t}} \quad . \tag{3.13}$$

Furthermore, we know that at finite momentum $\mathbf{p}$, both the eigenvalue λ and $\mathbf{p}^2$ appear additively in the eigenvalue problem. Scaling the momentum with a length scale ξ, we see that the instability signaled by the appearance of a zero eigenvalue as $t \to 0$ is accompanied by a diverging length scale

$$\xi(t \to 0) \propto (\text{const.} \times t)^{-1/4\sqrt{t}} \quad . \tag{3.14a}$$

Clearly, ξ is the correlation length for the phase transition that is the result of the instability of the normal metal phase. According to the usual definition of the correlation length exponent ν, $\xi \propto t^{-\nu}$, one has

$$\nu = \infty \quad . \tag{3.14b}$$

We now turn to the order parameter susceptibility and the critical exponent γ. Since the diverging correlation length ξ appears in the particle-particle spin-triplet channel, we expect the order parameter to be one of the corresponding components of the Q matrix, e.g. ${}^1_1Q$. The order parameter susceptibility will then be the corresponding two-point correlation function. In the symbolic notation of Sec. III A, the susceptibility is given by

$$\begin{aligned}\chi(\Omega_n) &= (\delta_n|M^{-1}|\delta_n) \\ &\equiv \sum_{1,2,3,4} \delta_{1+2,n}\, M^{-1}_{12,34}\, \delta_{3+4,n} \quad .\end{aligned} \tag{3.15}$$

Now consider the eigenvalue equation for the matrix M, Eq. (3.2), and go into the basis of eigenfunctions $f^{(i)}$ to the eigenvalues λ_i. Inserting two complete sets of eigenfunctions in Eq. (3.15) we obtain

$$\chi(\Omega_n) = \sum_i (\delta_n|f^{(i)})\,\frac{1}{\lambda_i\,(f^{(i)}|f^{(i)})}\,(f^{(i)}|\delta_n) \quad . \tag{3.16}$$

Let λ be the smallest eigenvalue, and f_λ the corresponding eigenfunction. Then the leading contribution to the diverging susceptibility as $\lambda \to 0$ is obtained by keeping only f_λ in Eq. (3.16). At zero external frequency we obtain

$$\chi(\Omega=0)=\left(\int_{-\infty}^{\infty}d\omega\, f_\lambda(\omega)\right)^2/\lambda\int_{-\infty}^{\infty}d\omega\, f_\lambda^2(\omega)\quad . \tag{3.17}$$

The divergence of χ as $\lambda\to 0$ is determined by the ultraviolet behavior of the eigenfunction f. With Eq. (3.11) we obtain

$$\chi(\Omega=0)\propto t^{-1}\quad . \tag{3.18a}$$

This means that the critical exponent γ, defined by $\chi\propto t^{-\gamma}$, has its mean-field value,

$$\gamma=1\quad . \tag{3.18b}$$

Finally, we determine the critical wavenumber dependence of the order parameter susceptibility, i.e., the critical exponent η. Here the disorder part of the action makes a difference, and so we consider it explicitly. From Eqs. (3.4b) and (2.16a) we see that, with $a=\mathcal{D}^{-1}$, the eigenvalue λ as a function of both t and the momentum $\mathbf{p}$ has the form $\lambda\propto t^{1/2\sqrt{t}}+\text{const.}\times\mathbf{p}^2$. In a scaling sense,[6] we therefore have $t^{1/2\sqrt{t}}\sim\mathbf{p}^2$, or

$$t\sim\left(\frac{\ln\ln 1/\mathbf{p}^2}{\ln 1/\mathbf{p}^2}\right)^2\quad , \tag{3.19}$$

plus terms that are less leading for $\mathbf{p}\to 0$. Together with Eq. (3.18a) this implies for the critical susceptibility

$$\chi(\mathbf{p},t=0)\propto\left(\frac{\ln 1/\mathbf{p}^2}{\ln\ln 1/\mathbf{p}^2}\right)^2\quad . \tag{3.20a}$$

According to the definition of the critical exponent η, $\chi(\mathbf{p},t=0)\propto\mathbf{p}^{-(2-\eta)}$ this means

$$\eta=2\quad , \tag{3.20b}$$

up to logarithmic corrections to power-law critical behavior.

IV. THE ORDERED PHASE

A. The gap equation

We now turn to the ordered phase. Since the diverging susceptibility, Sec. III C, is in the particle-particle spin triplet channel, we know that the order parameter will be a Q matrix in that sector. The symmetry group in that sector is $U(1)\times SU(2)$ (the usual gauge symmetry plus rotations in spin space), which is spontaneously broken to $Z_2\times U(1)$. We choose $r=s=1$ as the direction in which the symmetry is broken, and accordingly make an *ansatz*[13]

$$ {}^{i}_{r}Q_{12}(\mathbf{x})\Big|_{\rm sp}=\delta_{n_1,-n_2}\,\delta_{r1}\,\delta_{i1}\,Q_{n_1}\quad , \tag{4.1a}$$

$$ {}^{i}_{r}\widetilde{\Lambda}_{12}(\mathbf{x})\Big|_{\rm sp}=\delta_{n_1,-n_2}\,\delta_{r1}\,\delta_{i1}\,\Lambda_{n_1}\quad , \tag{4.1b}$$

for the saddle-point values of the fields in the broken-symmetry phase. Using this *ansatz* in the saddle-point equations, Eqs. (2.9), yields the following equations for Λ_n and Q_n,

$$Q_n=-\frac{1}{2}\,\Lambda_{-n}\,\frac{1}{V}\sum_{\mathbf{k}}\frac{1}{\omega_n^2+\xi_{\mathbf{k}}^2-(\Lambda_n)^2}\quad , \tag{4.2a}$$

$$\Lambda_n=\frac{\pi T}{N_{\rm F}}\sum_m K_{nm}\,\frac{1}{V}\sum_{\mathbf{k}}\frac{\Lambda_m}{\omega_m^2+\xi_{\mathbf{k}}^2-(\Lambda_m)^2}\quad . \tag{4.2b}$$

Here

$$K_{nm}=y\,\ln\left|\frac{n-m}{n+m}\right|\quad , \tag{4.2c}$$

and we have used the fact that $\Lambda_n=-\Lambda_{-n}$ is an odd function of the frequency.

It is clear from Eq. (2.7a) (see the third term on the r.h.s of that equation) that Λ_n plays the role of the self energy in the particle-particle spin-triplet channel or of the gap function for the triplet superconductor. A crucial question now arises about the reality properties of Λ_n. It follows from Eq. (4.2a) that Λ_n is real (imaginary) if and only if Q_n is real (imaginary). According to the formal symmetry properties, Eqs. (2.6d), (2.6e), Q_n should be real. However, as we have discussed at the end of Sec. II B, this is misleading. In order for the field theory to be stable and correctly reproduce perturbation theory, the integration contour for the Q must be deformed, and for our saddle point to lie on the contour we must choose Q_n, and hence Λ_n, to be imaginary. Another requirement is of course that this choice minimizes the free energy. We will show in the next subsection that this is indeed the case. We thus identify $\Delta_n=-i\Lambda_n$, with Δ_n real,[14] as the gap function, and obtain the gap equation in the form

$$\Delta_n=\frac{\pi T}{N_{\rm F}}\sum_m K_{nm}\,\frac{1}{V}\sum_{\mathbf{k}}\frac{\Delta_m}{\omega_m^2+\xi_{\mathbf{k}}^2+(\Delta_m)^2}\quad . \tag{4.3}$$

Note that this gap equation has the same structure as the one in BCS theory (except for the frequency dependences of the kernel and the gap function), and that it is identical to the gap equation that was derived in Ref. 4.[15] It was shown in that reference that there are nonzero solutions for Δ_n for $y>y_c=0.25$, i.e. in the region in parameter space where we have found the normal metal to be unstable. We also note that the gap equation is the same irrespective of whether or not we explicitly keep the disorder term in the action. This is the triplet analog of Anderson's theorem.

The gap equation, Eq. (4.3) was solved in Ref. 4, both analytically in certain limits and numerically for all frequencies, and there is no need to repeat this. Let us discuss, however, one important point that was not mentioned in the earlier work. To this end, we first recall that by using the approximation, Eq. (3.6), for the kernel of the integral equation, Eq. (4.3), one can transform

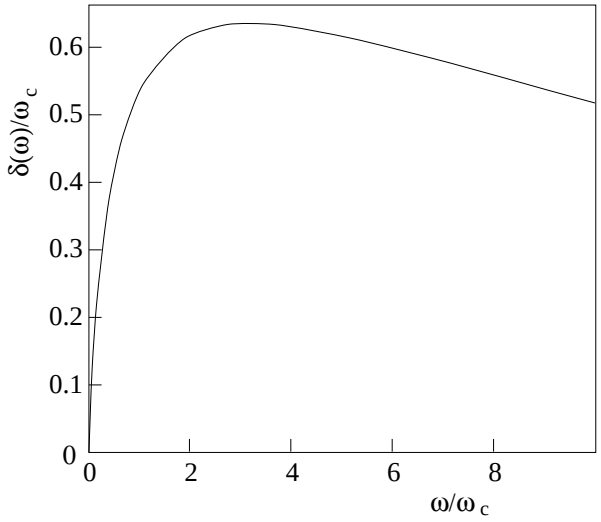

FIG. 1. Low-frequency behavior of the gap function for $y = 1.3\, y_c$. See the text for an explanation of the numerical procedure.

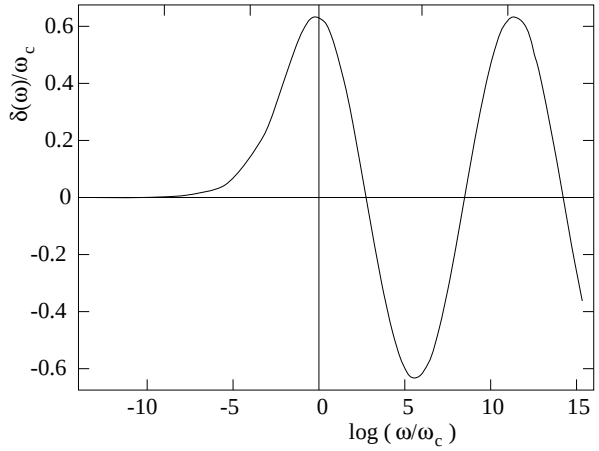

FIG. 2. Overall behavior of the gap function for $y = 1.3\, y_c$ on a logarithmic frequency scale. See the text for an explanation of the numerical procedure.

the integral equation into an ODE, as we did for the eigenfunction in Sec. III B. We briefly recall the most important features of the solution. On the imaginary axis, $\Delta_n = \Delta(i\omega_n) \equiv \delta(\omega_n)$, and the limiting behavior of $\delta(\omega)$ for asymptotically small ω is

$$\delta(\omega \to 0) = -2\, y\, \omega \ln(\omega/\omega_c) \quad , \tag{4.4a}$$

with ω_c a frequency scale that may depend on y, and for asymptotically large frequencies

$$\delta(\omega \to \infty) = \text{const.} \times \cos\left(|t|^{1/2} \ln \omega + \phi\right) \quad , \tag{4.4b}$$

with a constant prefactor and a phase ϕ that we could not determined analytically. Figure 1 shows the function $\delta(\omega)$ for small ω, and Fig. 2 shows the oscillatory behavior at large ω on a logarithmic scale. For this numerical solution at a fixed value of y, Eq. (4.4a) was used to provide an initial condition for a very small but non-zero frequency, and then the ODE was iterated towards larger frequencies. For the initial condition, a value of ω_c was chosen arbitrarily. It turns out that one finds a solution irrespective of the value of ω_c chosen, and that all of these solutions are qualitatively the same. This behavior of the numerics is readily understood by inspecting the gap equation. It is easy to see that if $\delta(\omega)$ is a solution, then so is $\gamma(\omega) = b^{-1}\, \delta(b\omega)$ with an arbitrary positive number b. This scaling property holds for both the integral equation, Eq. (4.3), and for the ODE that is derived from it by approximating the kernel. This raises the question of what determines the scale ω_c in Eq. (4.4a). In principle, the answer must be inherent in the gap equation itself, but we can obtain ω_c from a much simpler argument. As we have seen in Sec. III, the transition is an ordinary continuous phase transition with a diverging length scale, so scaling will work. We further know that frequencies scale like wavenumbers squared, $\omega \sim \mathbf{p}^2$. From a scaling point of view, ω_c is the critical frequency scale, and hence

$$\omega_c(t) \propto \xi^{-2}(t) \propto |t|^{1/2\sqrt{t}} \quad , \tag{4.5}$$

where we have used Eq. (3.14a). This means that $\delta_{t\to 0}(\omega) \to 0$ for all ω, as one would expect, and the overall amplitude of the order parameter vanishes like ω_c. For the critical exponent β, defined by $\Delta \propto t^\beta$, this implies

$$\beta = 2\nu = \infty \quad . \tag{4.6}$$

B. The free energy

We now check whether the saddle-point solution that we found to exist is stable, i.e. whether it has a free energy that is lower than that of the normal metal state. In mean-field approximation, the free energy is given by the action, with the fields replaced by their mean-field values. By substituting Eqs. (4.1) in Eq. (2.7a) and expanding and resumming the Tr ln term, we find for the free energy density in mean-field approximation

$$f = -2T \sum_n \Lambda_n\, Q_n - T \sum_n \frac{1}{V} \sum_{\mathbf{k}} \ln\left[1 - \Lambda_n^2/(\omega_n^2 + \xi_{\mathbf{k}}^2)\right] , \tag{4.7}$$

where we have used Eqs. (4.2) to rewrite the term that was quadratic in Q_n as a term that is bilinear in Q_n and Λ_n. We next use Eqs. (4.2) again to make this term quadratic in Λ_n,

$$f = \frac{-N_{\rm F}}{\pi}\, T \sum_{n,m} \Lambda_n\, K^{-1}_{nm}\, \Lambda_m$$
$$-T \sum_n \frac{1}{V} \sum_{\mathbf{k}} \ln\left[1 - \Lambda_n^2/(\omega_n^2 + \xi_{\mathbf{k}}^2)\right] \quad , \qquad (4.8a)$$

where K^{-1} is the inverse of the operator K, Eq. (4.2c), in the sense that

$$T \sum_{n'} K^{-1}_{nn'}\, K_{n'm} = \delta_{nm} \quad . \qquad (4.8b)$$

Now we use $\Lambda_n = -i\Delta_n$, and expand in powers of the order parameter. This yields the Landau expansion

$$f = T^2 \sum_{n,m} \Delta_n\, t_{nm}\, \Delta_m + T \sum_n u_n\, \Delta_n^4 + O(\Delta^6) \quad , \qquad (4.9a)$$

where

$$t_{nm} = \frac{N_{\rm F}}{\pi T}\, K^{-1}_{nm} - \frac{1}{T}\, \delta_{nm}\, \frac{1}{V} \sum_{\mathbf{k}} \frac{1}{\omega_n^2 + \xi_{\mathbf{k}}^2} \quad , \qquad (4.9b)$$

and

$$u_n = \frac{1}{2V} \sum_{\mathbf{k}} \frac{1}{(\omega_n^2 + \xi_{\mathbf{k}}^2)^2} > 0 \quad . \qquad (4.9c)$$

u_n is positive definite, and hence the theory is stable. Furthermore, t_{nm} is negative for small K^{-1} or large K, i.e. for strong coupling. Hence the free energy in a state with a nonzero order parameter is indeed lower than in a state with zero order parameter, provided that the coupling is sufficiently strong, i.e. $y > y_c$. All of these considerations are in exact analogy to the case of BCS theory. We note that if Λ_n were real, then the sign of the quadratic term in Eq. (4.9a) would be reversed, and consequently the free energy would favor order at *small* coupling, which is somewhat paradoxical. This was actually mentioned already in Berezinskii's paper,[1] and later investigated in more detail by Heid.[16]

C. Linear response in the ordered phase

We now calculate the linear response in the ordered phase, specifically the magnetic susceptibility and the electrical conductivity, in Gaussian approximation. This will establish that the ordered phase is really a superconductor.

To start with, we need the saddle-point Green function. It's inverse is (see Eq. (2.7a))

$$G^{-1} = G_0^{-1} - i\widetilde{\Lambda} \quad . \qquad (4.10)$$

Using Eq. (4.1b), and inverting the 2×2 matrix, we obtain

$$G_{nm}(\mathbf{k}) = \delta_{nm}\, \mathcal{G}_n(\mathbf{k})\, (\tau_0 \otimes s_0) + \delta_{n,-m}\, \mathcal{F}_n(\mathbf{k})\, (\tau_1 \otimes s_1) \ , \qquad (4.11a)$$

where

$$\mathcal{G}_n(\mathbf{k}) = \frac{-(i\omega_n + \xi_{\mathbf{k}})}{\omega_n^2 + \xi_{\mathbf{k}}^2 + \Delta_n^2} \quad , \qquad (4.11b)$$

$$\mathcal{F}_n(\mathbf{k}) = \frac{-\Delta_n}{\omega_n^2 + \xi_{\mathbf{k}}^2 + \Delta_n^2} \quad . \qquad (4.11c)$$

Notice again the structural analogy to BCS theory.

1. The anomalous susceptibility

Before we calculate observable correlation functions, let us focus on the Goldstone modes. Although not directly observable, they are of crucial importance for the former. As we already noted in Sec. IV A, the relevant symmetry group is $U(1) \times SU(2)$, which is spontaneously broken to the little group $Z_2 \times U(1)$. The dimension of the relevant quotient space is dim $(U(1) \times SU(2)/U(1)) = 5$, so there are five Goldstone modes. With our choice of the order parameter in the sector $r = s = 1$, they are represented by the correlation functions

$$T \sum_{1,2} \left\langle {}^i_r Q_{1+n,-1}(\mathbf{k})\, {}^i_r Q_{2+n,-2}(-\mathbf{k}) \right\rangle \quad , \qquad (4.12)$$

with $r = 1; i = 1, 2$, or $r = 2; i = 1, 2, 3$. It will turn out that the $r = 2, i = 1$ Goldstone mode couples to the density response, so let us focus on the anomalous susceptibility

$$\chi^{(a)}_{12,34}(\mathbf{k}) = \left\langle {}^1_2 Q_{1,2}(\mathbf{k})\, {}^1_2 Q_{3,4}(-\mathbf{k}) \right\rangle \quad . \qquad (4.13)$$

Now consider the eigenvalue problem

$$\sum_{3,4} \chi^{(a)}_{12,34}\, f_{34} = \lambda\, f_{12} \quad . \qquad (4.14)$$

This is the same eigenvalue problem as the one we considered in Sec. III in the symmetric phase. We therefore know that at zero momentum there is an eigenvalue λ_0 that is infinite everywhere in the broken symmetry phase. We further know (see Sec. III C) that the wavenumber dependence of λ^{-1} quadratic. Perturbation theory can be used to convince oneself that the frequency dependence is also quadratic, although determining the prefactors would be hard. Dropping all prefactors, we thus have the structure

$$\lambda(\mathbf{p}, \Omega) = \frac{1}{\mathbf{p}^2 + \Omega^2} \quad , \qquad (4.15)$$

everywhere in the ordered or broken symmetry phase.

2. The transverse current susceptibility

Let us now calculate the transverse current susceptibility, which determines the magnetic susceptibility and hence the Meissner effect. By adding an appropriate source term to our action, we obtain for the transverse susceptibility in saddle-point approximation

$$\chi_{\rm T}(\mathbf{k}\to 0) = \frac{1}{m^2}\sum_{\mathbf{p}} \mathbf{p}^2\, T\sum_n \left[-\mathcal{G}_n(\mathbf{p})\,\mathcal{G}_n(\mathbf{p}) + \mathcal{F}_n(\mathbf{p})\,\mathcal{F}_{-n}(\mathbf{p})\right] \quad , \quad (4.16)$$

in $D = 2$ dimensions. Notice that in a diagrammatic language, Eq. (4.16) is a simple bubble. In the transverse channel, this is sufficient since all vertex corrections vanish. In the language of the present field theory, saddle-point and Gaussian approximations coincide. Now we do the integrals in Eq. (4.16). A convenient way to do this is to put an upper cutoff on the integral over $\xi_{\mathbf{p}}$, do that integral first, and let the cutoff go to infinity in the end. One can readily check that this procedure yields the correct answer in the BSC case (where convergence problems can be avoided by doing the frequency integration first). We then find[2]

$$\chi_{\rm T}(\mathbf{k}\to 0) = \frac{n}{m}\left[1 - \int_0^\infty d\omega\, \frac{\delta^2(\omega)}{\left(\omega^2+\delta^2(\omega)\right)^{3/2}}\right] \quad . \quad (4.17)$$

The integral is positive definite, therefore $\chi_{\rm T} < n/m$ and we have an ordinary Meissner effect. Notice that the reality properties of Λ and Δ are once again crucial for this conclusion: If Λ in the particle-particle spin-triplet channel were real, then the correction to the free fermions result in Eq. (4.16) would be positive! We also note that $\delta(\omega)$ as given by Eq. (4.4a) renders the frequency integral in Eq. (4.16) infinite. This is an artifact of the theory that is due to a missing self-consistency. As was discussed in Ref. 2, taking into account the feedback of the nonzero order parameter on the electron-electron interaction would cure this problem.

3. The longitudinal response

The calculation of the longitudinal current, or density, response, and hence the conductivity, is much more involved. The reason is that in order to obtain a conserving approximation, one needs to solve the full Gaussian theory (in diagrammatic language, one needs to take the vertex corrections into account). This is well known in BCS theory,[17] and the calculation for the present case proceeds analogously. As a check of our procedure, we have also performed the calculation for the BCS case and have reproduced the standard results.

We start out by expanding the action, Eq. (2.7a), to Gaussian order about the saddle-point solution, Eqs. (4.2). For the part that is quadratic in $\delta\widetilde{\Lambda} = \widetilde{\Lambda} - \widetilde{\Lambda}_{\rm sp}$ we find

$$\mathrm{Tr}\left(G_{\rm sp}\,\delta\widetilde{\Lambda}\,G_{\rm sp}\,\delta\widetilde{\Lambda}\right) = 4\sum_{1,2,3,4}\frac{1}{V}\sum_{\mathbf{k}}\sum_{r,s}\sum_{i,j}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) \times {}^{ij}_{rs}\widetilde{A}_{12,34}(\mathbf{k})\,{}^{j}_{s}(\delta\widetilde{\Lambda})_{34}(-\mathbf{k}) \quad . \quad (4.18a)$$

Here

$$\begin{aligned}{}^{ij}_{rs}\widetilde{A}_{12,34}(\mathbf{k}) = {}&\delta_{13}\,\delta_{24}\,\varphi^{00}_{12}(\mathbf{k})\,M^{00}_{rs}\,\delta_{ij} \\ &+\delta_{13}\,\delta_{2,-4}\,\varphi^{01}_{12}(\mathbf{k})\,M^{01}_{rs}\,m^{01}_{ij} \\ &+\delta_{1,-3}\,\delta_{24}\,\varphi^{10}_{12}(\mathbf{k})\,M^{10}_{rs}\,m^{10}_{ij} \\ &+\delta_{1,-4}\,\delta_{2,-3}\,\varphi^{11}_{12}(\mathbf{k})\,M^{11}_{rs}\,M^{11}_{ij} \quad , \end{aligned} \quad (4.18b)$$

with 4×4 matrices

$$\begin{aligned} M^{00} &= \begin{pmatrix} i\tau_3 & 0 \\ 0 & -i\tau_3 \end{pmatrix} \quad , \quad M^{01} = \begin{pmatrix} -i\tau_1 & 0 \\ 0 & -i\tau_1 \end{pmatrix} \quad , \\ M^{10} &= \begin{pmatrix} -i\tau_1 & 0 \\ 0 & i\tau_1 \end{pmatrix} \quad , \quad M^{11} = \begin{pmatrix} -i\tau_3 & 0 \\ 0 & -\tau_0 \end{pmatrix} \quad , \\ m^{01} &= \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} \quad , \quad m^{10} = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix} \quad . \end{aligned} \quad (4.18c)$$

and

$$\varphi^{00}_{nm}(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{p}}\mathcal{G}_n(\mathbf{p})\,\mathcal{G}_m(\mathbf{p}+\mathbf{k}) \quad , \quad (4.18d)$$

and φ^{01}, φ^{10}, and φ^{11} defined analogously with $\mathcal{G}\mathcal{G}$ in Eq. (4.18d) replaced by $\mathcal{G}\mathcal{F}$, $\mathcal{F}\mathcal{G}$, and $\mathcal{F}\mathcal{F}$, respectively.

Similarly, the term that couples $\delta\widetilde{\Lambda}$ and δQ can be written

$$\mathrm{Tr}\left(\delta\widetilde{\Lambda}\,\delta Q\right) = 4\sum_{1,2,3,4}\frac{1}{V}\sum_{\mathbf{k}}\sum_{r,s}\sum_{i,j}{}^{i}_{r}(\delta\widetilde{\Lambda})_{12}(\mathbf{k}) \times {}^{ij}_{rs}\widetilde{B}_{12,34}(\mathbf{k})\,{}^{j}_{s}(\delta Q)_{34}(-\mathbf{k}) \ , \quad (4.19a)$$

where

$${}^{ij}_{rs}\widetilde{B}_{12,34}(\mathbf{k}) = \delta_{13}\,\delta_{24}\,\delta_{rs}\begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \delta_{ij} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i \quad . \quad (4.19b)$$

The full Gaussian action is

$$\mathcal{A}_{\rm G} = \mathrm{Tr}\left(G_{\rm sp}\,\delta\widetilde{\Lambda}\,G_{\rm sp}\,\delta\widetilde{\Lambda}\right) + \mathrm{Tr}\left(\delta\widetilde{\Lambda}\,\delta Q\right) + \mathcal{A}^{\rm p-p,t}_{\rm int}[\delta Q] \ . \quad (4.20)$$

This Gaussian action is too complicated to be handled conveniently, and approximations are necessary. In order to make sensible approximations, we recall that our purpose is to calculate the density response in a conserving approximation. In order to do so, it is crucial to

preserve the structure of the soft modes in the theory, while massive modes can be dealt with in very crude approximations. It turns out that the soft mode structure is preserved by keeping only the imaginary parts of the fields in the $r = 1, 2, 3$ channels, and only the real parts in the $r = 0$ channel, and integrating out the remaining parts in saddle-point approximation. This amounts to making the approximations

$$\begin{aligned} {}^{i}_{r}(\delta\widetilde{\Lambda})_{12} &\approx \frac{1}{2}\left[{}^{i}_{r}(\delta\widetilde{\Lambda})_{12} + \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r {}^{i}_{r}(\delta\widetilde{\Lambda})_{-1,-2} \right] \\ &= \begin{pmatrix} i\mathrm{Im} \\ i\mathrm{Im} \\ i\mathrm{Im} \\ \mathrm{Re} \end{pmatrix}_r {}^{i}_{r}(\delta\widetilde{\Lambda})_{12} \equiv {}^{i}_{r}\lambda_{12} \quad , \end{aligned} \tag{4.21a}$$

and approximating δQ by an analogously defined object q. λ and q have the symmetry properties

$${}^{i}_{r}\lambda_{12} = \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r {}^{i}_{r}\lambda_{-1,-2} \quad , \tag{4.21b}$$

$${}^{i}_{r}q_{12} = \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r {}^{i}_{r}q_{-1,-2} \quad , \tag{4.21c}$$

The net effect of these approximations is a symmetrization of the action with respect to positive and negative frequency values.

We now formally integrate out $\delta\widetilde{\Lambda}$. This yields the action entirely in terms of q,

$$\begin{aligned} \mathcal{A}_{\mathrm{G}}[q] = \frac{-4}{V} \sum_{\mathbf{p}} \sum_{1,2,3,4} \sum_{r,s} \sum_{i.j} {}^{i}_{r}q_{12}(\mathbf{p}) \\ \times \left[{}^{ij}_{rs}A^{-1}_{12,34}(\mathbf{p}) - {}^{ij}_{rs}B_{12,34} \right] {}^{j}_{s}q_{34}(-\mathbf{p}) \quad . \end{aligned} \tag{4.22a}$$

Here ${}^{ij}_{rs}A^{-1}_{12,34}$ is the inverse of the matrix

$$\begin{aligned} {}^{ij}_{rs}A_{12,34} = \frac{1}{4} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_j \Bigg[\begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_s {}^{ij}_{rs}\widetilde{A}_{12,34} \\ + \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_s {}^{ij}_{rs}\widetilde{A}_{-1,-2;3,4} \\ + \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_s {}^{ij}_{rs}\widetilde{A}_{1,2;-3,-4} \\ + \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_s {}^{ij}_{rs}\widetilde{A}_{-1,-2;-3,-4} \Bigg] \quad , \end{aligned} \tag{4.22b}$$

and

$${}^{ij}_{rs}B_{12,34} = \delta_{rs}\,\delta_{ij} \begin{pmatrix} 0 \\ + \\ + \\ 0 \end{pmatrix}_r \begin{pmatrix} 0 \\ + \\ + \\ + \end{pmatrix}_i B_{12,34} \quad , \tag{4.22c}$$

with

$$B_{12,34} = -\frac{1}{4}\,\pi N_{\mathrm{F}}\,T\,\delta_{1+2,3+4}\,\widetilde{K}_{12,34} \quad . \tag{4.22d}$$

By adding an appropriate source term to the action it is easily checked that the expression for the density susceptibility χ in terms of the q is the same as in terms of the Q, viz.

$$\chi(\mathbf{k}, \Omega_n) = 16T \sum_{1,2} \sum_{r=0,3} \left\langle {}^{0}_{r}q_{1+n,1}(\mathbf{k})\, {}^{0}_{r}q_{2+n,2}(-\mathbf{k}) \right\rangle \quad . \tag{4.23a}$$

From the expression of Q in terms of the fermion fields, Eq. (2.2), it is easy to see that the contributions to Eq. (4.23a) from $r = 0$ and $r = 3$ are identical, except at zero external frequency. We can therefore write

$$\chi(\mathbf{k}, \Omega_n \neq 0) = 32T \sum_{1,2} \left\langle {}^{0}_{3}q_{1+n,1}(\mathbf{k})\, {}^{0}_{3}q_{2+n,2}(-\mathbf{k}) \right\rangle \quad , \tag{4.23b}$$

and obtain the zero frequency susceptibility from Eq. (4.23b) in the limit $\Omega_n \to 0$.[18] Combining Eqs. (4.23b) and (4.22a) we have

$$\chi(\mathbf{k}, \Omega_n \neq 0) = 4T \sum_{1,2} {}^{00}_{33}M^{-1}_{1+n,1;2+n,2}(\mathbf{k}) \quad , \tag{4.24a}$$

where

$$M^{-1} = \left(A^{-1} - B \right)^{-1} \quad . \tag{4.24b}$$

Instead of inverting the matrix M directly, it is convenient to rewrite Eq. (4.24b) as an integral equation,

$$M^{-1} = A + A\,B\,M^{-1} \quad . \tag{4.25}$$

Ignoring the frequency indices for the time being, we need the matrix element

$${}^{00}_{33}M^{-1} = {}^{00}_{33}A + {}^{01}_{32}A\,B\,{}^{10}_{23}M^{-1} \quad , \tag{4.26}$$

where we have used the structure of B, Eq. (4.22c). ${}^{10}_{23}M^{-1}$ in turn obeys the integral equation

$$\begin{aligned} {}^{10}_{23}M^{-1} &= {}^{10}_{23}A + {}^{11}_{22}A\,B\,{}^{10}_{23}M^{-1} \\ &= {}^{10}_{23}A + {}^{11}_{22}A\,\beta\,{}^{10}_{23}A \quad , \end{aligned} \tag{4.27a}$$

where β is the solution of

$$\beta = B + B\,{}^{11}_{22}A\,\beta \quad . \tag{4.27b}$$

β is related to the anomalous susceptibility $\chi^{(a)}$, Eq. (4.13), by

$$\beta = {}^{11}_{22}A^{-1}\,\chi^{(a)}\,B \quad , \tag{4.27c}$$

with our approximation for $\chi^{(a)}$ that replaces Q in Eq. (4.13) by q.

We are now in a position to calculate the density susceptibility. Defining a vector $|\delta_n) \equiv \delta_{1,2-n}$ we can write χ, Eq. (4.24a), as a matrix element

$$\chi(\mathbf{k}, \Omega_n \neq 0) = 4T\,(\delta_n|\,{}^{00}_{33}M^{-1}(\mathbf{k})\,|\delta_n) \quad . \tag{4.28}$$

Let us first consider the 'bubble contribution',

$$\chi_0(\mathbf{k}, \Omega_n) = 4T\,(\delta_n|\,{}^{00}_{33}A(\mathbf{k})\,|\delta_n) \quad . \tag{4.29a}$$

Equations (4.22b) and (4.18) yield

$$\begin{aligned}\chi_0(\mathbf{k}, \Omega_n) &= T\sum_1 \big[\big(\varphi^{00}_{1+n,1}(\mathbf{k}) - \varphi^{11}_{1+n,1}(\mathbf{k})\big) \\ &\qquad + \big(\varphi^{00}_{1-n,1}(\mathbf{k}) - \varphi^{11}_{1-n,1}(\mathbf{k})\big)\big] \\ &\equiv I_1(\mathbf{k}, \Omega_n) \quad .\end{aligned} \tag{4.29b}$$

Doing the integrals in the limit of small wavenumber and frequency yields

$$I_1(\mathbf{k}, \Omega_n) = N_{\rm F} + O(\mathbf{k}^2, \Omega) \quad . \tag{4.29c}$$

Notice that the bubble contribution is not a conserving approximation for χ, as the susceptibility does not vanish at zero wavenumber. As in the BCS case, particle number conservation or gauge invariance will be restored by the coupling of the density fluctuations to a soft or massless Goldstone mode, as we now proceed to show.

We now consider the 'soft' part of χ, i.e. the one that couples to the Goldstone mode. From Eqs. (4.28), (4.27) we find

$$\begin{aligned}\chi_{\rm soft}(\mathbf{k}, \Omega_n) &= (\delta_n|\,{}^{01}_{32}A(\mathbf{k})\,B\,\chi^{(a)}(\mathbf{k})\,B\,{}^{10}_{23}A(\mathbf{k})|\delta_n) \\ &= \sum_{f,f'}(\delta_n|\,{}^{01}_{32}A(\mathbf{k})\,B|f)\,(f|\chi^{(a)}(\mathbf{k})|f') \\ &\qquad \times (f'|B\,{}^{10}_{23}A(\mathbf{k})|\delta_n) \quad ,\end{aligned} \tag{4.30a}$$

where we have inserted two complete sets $\{|f)\}$ and $\{|f')\}$ of eigenfunctions of $\chi^{(a)}$. Since we are interested in the coupling to the soft mode, we need to keep only the eigenfunction f whose eigenvalue λ is infinite at zero momentum and frequency, see Eq. (4.15). We thus have

$$\chi_{\rm soft}(\mathbf{k}, \Omega_n) = (\delta_n|\,{}^{01}_{32}A(\mathbf{k})|g)\,\lambda(\mathbf{k}, \Omega_n)\,(g|{}^{10}_{23}A(\mathbf{k})|\delta_n) \quad , \tag{4.30b}$$

with $|g) = B|f)$ the transformed eigenfunction from Eq. (3.3). The structure of the eigenvalue we know, Eq. (4.15), and what remains to be done is to calculate the coupling integral. Using the symmetry property $g_{12} = -g_{21}$ of the eigenfunction that follows from Eq. (3.4a), we find

$$\begin{aligned}I_2(\mathbf{k}, \Omega_n) &\equiv (\delta_n|\,{}^{01}_{32}A(\mathbf{k})|g) = (g|{}^{10}_{23}A(\mathbf{k})|\delta_n) \\ &= \frac{1}{2}\sum_1 \big[\big(\varphi^{01}_{1+n,1}\,g_{1+n,-1} - \varphi^{01}_{1-n,1}\,g_{1-n,-1}\big) \\ &\qquad + \big(\varphi^{01}_{1+n,1}\,g_{1+n,-1} - \varphi^{01}_{1-n,1}\,g_{1-n,-1}\big)\big] \quad .\end{aligned} \tag{4.31a}$$

Using the above symmetry property of g, and the fact that the anomalous Green function $\mathcal{F}$, Eq. (4.11c), is an antisymmetric function of the frequency, we see that I_2 is also an odd function of its frequency argument. A Taylor expansion yields

$$I_2(\mathbf{k}, \Omega_n) = -I_2(\mathbf{k}, -\Omega_n) \propto \Omega_n \quad . \tag{4.31b}$$

We conclude that the density susceptibility at small frequencies and wavenumbers has the structure

$$\chi(\mathbf{k}, \Omega_n) = N_{\rm F}\left(1 - \frac{a\,\Omega_n^2}{b\,\mathbf{k}^2 + c\,\Omega_n^2}\right) \quad , \tag{4.32a}$$

where a, b, c are constants that we have not determined explicitly. Since only the coupling to the Goldstone mode can restore the property $\chi(\mathbf{k} \to 0, \Omega_n) = 0$, we must have $a = c$. We have performed the analogous procedure for the BCS case, where $a = c$ is easily confirmed explicitly. We conclude that the structure of the density susceptibility is the same as in the BCS case, viz.

$$\chi(\mathbf{k}, \Omega_n) = N_{\rm F}\,\frac{(v\mathbf{k})^2}{\Omega_n^2 + (v\mathbf{k})^2} \quad , \tag{4.32b}$$

with v the velocity of the sound-like mode that is the analog of the Anderson-Bogolubov mode in BCS superconductors.[19] Here we have not determined v explicitly.

The electrical conductivity σ is determined by χ via

$$\sigma(\Omega_n) = e^2 \lim_{\mathbf{k}\to 0}\frac{\Omega_n}{\mathbf{k}^2}\,\chi(\mathbf{k}, \Omega_n) \quad . \tag{4.33}$$

Combining Eqs. (4.32b) and (4.33), we see that $\sigma(\Omega_n) \propto 1/\Omega_n$, and the real part of the conductivity as a function of real frequencies has a delta-function contribution

$$\text{Re}\;\sigma(\Omega) = \frac{e^2\,N_{\rm F}\,v^2}{\pi}\,\delta(\Omega) \quad . \tag{4.34}$$

The ordered phase is thus a true superconductor. If we neglect the explicit disorder term in the action, then the delta function in Eq. (4.34) will exhaust the f-sum rule, which in turn determines the velocity v. However, with the disorder term taken into account, there is a continuous spectrum in addition to the delta function, and the determination of v would require an explicit calculation of the prefactors in Eq. (4.32a).

V. DISCUSSION AND CONCLUSION

A. Summary of results

Let us summarize. We have considered in detail the consequences of the attractive interaction in the particle-particle spin-triplet channel that has been predicted to exist in 2-D disordered interacting electron systems.[4] We

have found that this interaction leads to an instability of any normal metal phase in the system. The instability is characterized by a diverging length scale ξ, and has all the characteristics of a continuous quantum phase transition. The critical exponents ν, γ, and η are

$$\nu = \infty \quad , \quad \gamma = 1 \quad , \quad \eta = 2 \quad , \tag{5.1a}$$

where the infinite value of ν indicates an essential singularity in the dependence of ξ on the distance from criticality t,

$$\xi \propto (c\,t)^{-1/4\sqrt{t}} \quad , \tag{5.1b}$$

with c a constant, and the value of η is to be understood as a complicated logarithmic dependence of the critical order parameter susceptibility on the wavenumber, Eq. (3.20a).

This instability identifies the order parameter of the phase transition, viz. an anomalous density in the particle-particle spin-triplet channel. Approaching the transition from the ordered phase, the order parameter critical exponent is

$$\beta = 2\nu \quad , \tag{5.2a}$$

in the sense that the amplitude of the frequency dependent order parameter vanishes like

$$\Delta \propto \xi^{-2} \quad . \tag{5.2b}$$

The phase transition thus has all the characteristics of an ordinary quantum critical point. The ordered phase has all the characteristics of a superconductor, including a Meissner effect and an infinite static electrical conductivity. While these conclusions are general, the detailed critical behavior, and in particular the values of the exponents, depend on our particular choice of the interaction kernel, Eq. (2.8c).

B. Discussion

As we have mentioned before, the reality properties of the gap function are crucial for the stability of our superconducting state. If the expectation value of ${}^1_1Q$ were real, then the sign of the $(\Delta_m)^2$ term in Eq. (4.3) would be negative. Irrespective of whether or not this modified gap equation has physical solutions, this would lead to a free energy that is minimized by a non-zero gap function at *weak* coupling. This was noticed already by Berezinskii,[1] who proposed that odd-gap superconductivity or superfluidity would appear in a temperature window. Later, a more detailed discussion of the free energy was given,[16] and it was shown that the different sign in the gap equation also leads to the Meissner kernel having the wrong sign.[20] This led to the abandonment of Berezinskii-type odd-gap proposals[3] in the context of high-T_c superconductivity. Instead, these authors then considered more complicated scenarios of composite order parameters that do not suffer from these problems.

In the present theory, these problems are solved by realizing that there is a choice with respect to the reality properties of the gap function we call Λ_n, and that the physical choice is to make it imaginary. Let us recall the observations that led to this realization. First of all, with the conventional choice of a real Λ_n, the field theory that results from expanding about a Fermi liquid saddle point is unstable in the particle-particle spin-triplet channel, which forces a deformation of the integration contour. The validity of this procedure was then checked by comparing the perturbation theory that results from the field theory with ordinary perturbation theory within the framework of a second quantized Hamiltonian. The comparison can be made in the very simple case of non-interacting electrons, where there is no question as to what the correct answer is. The perturbative result is also in quantitative agreement with experiment. Next, in order for the saddle point that describes the superconducting phase to lie on the deformed contour, the saddle-point fields, which correspond to anomalous expectation values in a fermionic formulation, must be chosen to be imaginary. This choice turns out to minimize the saddle-point free energy, while the conventional choice would maximize it. The conclusion is that our interpretation of $\langle {}^1_1Q_{n-1,-n}\rangle$ as imaginary is correct, and there is no ambiguity or freedom left. We note that the particle-particle spin-triplet channel is the only one where the determination of the reality properties of the expectation values is not straightforward.[21] In all other channels, the symmetry properties of the fluctuating fields, Eqs. (2.6), agree with the convergence requirements of the field theory, and no deformation of the contour is necessary. We do not know why the particle-particle spin-triplet channel is special in this respect.

One might object that the symmetry properties of the Q matrix, Eqs. (2.6), just encode the properties of the underlying fermionic fields, and must therefore also hold after taking expectation values, which would make $\langle {}^1_1Q_{n-1,-n}\rangle$ real. This objection is fallacious, however, due to the following reasons. The symmetry properties hold for the fluctuating fields, which are just dummy integration variables and have no direct physical meaning. Deforming the integration contour to apply the method of steepest descent is therefore permissible as long as one does not cross any singularities. This we have checked be comparing with fermionic perturbation theory. The expectation value $\langle {}^1_1Q_{n-1,-n}\rangle$, on the other hand, does have a physical meaning, but it does not follow from Eqs. (2.6) that it must be real. In the absence of spontaneous symmetry breaking, of course $\langle {}^1_1Q_{n-1,-n}\rangle = 0$. To obtain a non-zero expectation value, one must consider a small field that is conjugate to ${}^1_1Q_{n-1,-n}$. If this field were static, then hermiticity would indeed require $\langle {}^1_1Q_{n-1,-n}\rangle$ to be real. However, in the particle-particle spin-triplet channel the conjugate field cannot be static. The real-

ity properties of the anomalous expectation value then depend on the properties of the unphysical, time dependent, symmetry breaking field, which are not given a priori. They could be determined from the requirement that $\langle {}^1_1 Q_{n-1,-n} \rangle$ be imaginary, but we have not done that. We also note that our saddle point violates only the hermiticity requirement, Eq. (2.6e), but *not* the Pauli principle, Eq. (2.6d), in agreement with the above discussion.

One might wonder why the static conductivity is affected by the existence of a gap function, given that the latter is zero at zero frequency. The answer lies in the fact that the infinite conductivity, like the Meissner effect, is a direct consequence of the broken symmetry and the resulting Goldstone mode.[22] Indeed, in BCS theory with its constant gap function, the prefactor of the delta-function in the conductivity is independent of the gap. More specifically, given the existence of a sound-like Goldstone mode that couples to the density fluctuations, particle number conservation determines the form, Eq. (4.32b), of the density susceptibility, which in turn guarantees an infinite static conductivity. On an even more technical level, we observe that while the Goldstone susceptibility, Eq. (4.12), indeed vanishes at zero external frequency, the actual coupling to the density fluctuations is more complicated. From Eq. (4.31a) we see that the coupling is given by the more general susceptibility $\chi^{(a)}_{12,34}$, Eq. (4.13), multiplied by the eigenfunction g, which is a generalized gap function. The net result, as far as frequency dependences is concerned, is the same as for conventional superconductors.

We have neglected the explicit disorder term in the action, and have taken the quenched disorder into account only implicitly, by using the particle-particle spin-triplet interaction from Ref. 4. At the level of the Gaussian theory, keeping the disorder explicitly just complicates the calculations without leading to any essential changes, as we have pointed out throughout the paper. For instance, the gap equation is independent of the disorder (this is the spin-triplet analog of Anderson's theorem), and the critical behavior is qualitatively unchanged. Another (minor) modification is that the δ-function contribution to the conductivity no longer exhausts the f-sum rule, but rather sits on top of a continuous spectrum. This conclusion from the Gaussian theory would indeed be correct if we could work in dimensions $D > 2$. However, Eqs. (2.8) are valid only in $D = 2$, and in $D > 2$ the analogous interaction does not lead to a superconducting ground state.[4] In $D = 2$, on the other hand, the explicit disorder term in the action has more subtle, qualitative consequences, which we discuss next.

C. Outlook

Throughout this paper, we have ignored the well-known fact that, in a 2-D disordered interacting electron system at zero temperature, the paramagnetic normal metal phase is also unstable against the formation of an insulator, and a ferromagnetic phase. As a result, as we have stressed elsewhere,[23] the nature of the ground state in the absence of spin-flip scattering processes is not known, even in the absence of the attractive spin-triplet interaction discussed above.

We conclude that in a 2-D disordered electron fluid, there are at least three competing instabilities of the normal metal ground state. Which one of these wins is a priori unclear, and answering this question would require a detailed and consistent study of the free energies for the various phases, which has not been attempted so far. It is certainly conceivable that different instabilities dominate in different parameter regimes, leading to very different ground states. It is also possible that the tendencies towards forming an insulator and a spin-triplet even-parity superconductor, respectively, effectively cancel one another, leading to a normal metal-like phase at least in a certain temperature window. This last possibility is the basic idea behind recent proposals to explain the observed metal-insulator transition in Si MOSFETs and other 2-D electron systems in terms of our exotic superconductivity.[2] However, a better understanding of these issues will require much more work.

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APPENDIX A: WEAK-LOCALIZATION CORRECTIONS, AND THE STABILITY OF THE FIELD THEORY

The purpose of this appendix is to check that our field theory, defined by the action and the integration contour specified in Sec. II B, correctly reproduces the results of standard perturbation theory. For this purpose, we choose to calculate the well-known weak-localization corrections to the conductivity within our formalism.

To this end, we restore the disorder part of the action, $\mathcal{A}_{\rm dis}$, although we will not need it explicitly. It is well known that non-interacting disordered electrons in $D = 2$ dimensions in the absence of spin-orbit scattering do not have a metallic phase.[24] In perturbation theory, starting from the Boltzmann value for the conductivity σ and

performing a disorder expansion, this phenomenon manifests itself as corrections at first order that diverge logarithmically as the frequency Ω approaches zero. These corrections are of the form

$$\delta\sigma = \frac{1}{\epsilon_{\rm F}\tau}\,(a_{\rm s}+a_{\rm t})\,\ln\Omega\tau \quad . \tag{A1}$$

Here $\epsilon_{\rm F}$ is the Fermi energy, τ is the elastic scattering mean-free time, and the coefficients $a_{\rm s}$ and $a_{\rm t}$ are due to contributions from the particle-particle spin-singlet and spin-triplet channels, respectively. We need only the ratio of these two coefficients. In the absence of any spin-flip scattering processes as well as of magnetic fields, fermionic perturbation theory yields[25]

$$a_{\rm t} = -3\,a_{\rm s} \quad . \tag{A2}$$

This result has been confirmed by experiments on thin metallic films.[26] By deliberately contaminating the sample with a strong spin-orbit scatterer, the triplet contribution to the weak-localization effect can be eliminated, which leads to a sign change of the effect as well as a reduction of the amplitude by a factor of 1/2. Eq. (A2) is thus very well confirmed.

Perturbation theory within the Q-field theory[7] yields structurally

$$\begin{aligned}\delta\sigma &\propto \frac{-1}{V}\sum_{\mathbf{p}}\big[\big\langle {}^0_{-1}(\delta Q)(\mathbf{p})\,{}^0_1(\delta Q)(-\mathbf{p})\big\rangle\ {\rm tr}\,s_0 s_0\,{\rm tr}\,\tau_1\tau_1 \\ &\qquad +3\,\big\langle {}^1_1(\delta Q)(\mathbf{p})\,{}^1_1(\delta Q)(-\mathbf{p})\big\rangle\ {\rm tr}\,s_1 s_1\,{\rm tr}\,\tau_1\tau_1\big] \\ &\propto \frac{1}{V}\sum_{\mathbf{p}}\big[\big\langle {}^0_1(\delta Q)(\mathbf{p})\,{}^0_1(\delta Q)(-\mathbf{p})\big\rangle \\ &\qquad -3\,\big\langle {}^1_1(\delta Q)(\mathbf{p})\,{}^1_1(\delta Q)(-\mathbf{p})\big\rangle\big] \quad . \end{aligned} \tag{A3}$$

For notational simplicity we have suppressed the frequency labels on the Q matrices, which are related to the external frequency Ω. In the presence of disorder, the $\langle Q\,Q\rangle$ correlation functions are diffusive,[7] and therefore the momentum integral yields a $\ln\Omega\tau$ in $D=2$. Comparing Eqs. (A1) and (A3) we see that the field theory yields the same result as fermionic perturbation theory, provided that the two correlation functions have the same sign and magnitude. This is indeed the case with the procedure explained in Sec. II B, or, equivalently, if one ignores any convergence questions and formally performs the Gaussian integrals over Q, as was done in Ref. 7. We conclude that the field theory, with the particle-particle spin-triplet matrix elements integrated over the contour chosen in Sec. II B, correctly reproduces perturbation theory, which in turn is in quantitative agreement with experiment.

[1] V.L. Berezinskii, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 628 (1974) [JETP Lett. **20**, 287 (1974)].

[2] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **58**, 8214 (1998); D. Belitz and T.R. Kirkpatrick, Ref. 4.

[3] A. Balatsky and E. Abrahams, Phys. Rev. B **45**, 13125 (1992); E. Abrahams, A. Balatsky, J.R. Schrieffer, and P.B. Allen, Phys. Rev. B **47**, 513 (1993); E. Abrahams, A. Balatsky, D.J. Scalapino, and J.R. Schrieffer, Phys. Rev. B **52**, 1271 (1995).

[4] T.R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **66**, 1533 (1991); D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **46**, 8393 (1992).

[5] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, New York 1988).

[6] We use the notation $a \cong b$ for "a is isomorphic to b", $a \propto b$ for "a is proportional to b", and $a \sim b$ for "a scales like b".

[7] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B **56**, 6513 (1997); D. Belitz, F. Evers, and T.R. Kirkpatrick, Phys. Rev. B **58**, 9710 (1998).

[8] In Sec. IV, we will use ${}^1_1Q$ as the superconducting order parameter. If desirable, a different linear combination of $\psi\psi$ and $\bar\psi\bar\psi$, e.g. just a $\psi\psi$ or a $\bar\psi\bar\psi$ alone, could be chosen as the order parameter. This would simply amount to a rotation in the space of bilinear products of fermion fields. An order parameter with a structure similar to ${}^1_1Q$ appears in the Nambu formulation of BCS theory, see, e.g., J.R. Schrieffer, *Theory of Superconductivity*, Benjamin (Reading, Mass. 1983).

[9] In order to keep our discussion technically as simple as possible, we consider only a special case of the spin-triplet attraction derived in Ref. 4. Eqs. (2.8) represent the potential resulting from the spin-fluctuation mechanism discussed in that reference, in the limit of a large spin susceptibility.

[10] M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).

[11] For general matrix elements, it is in a nontrivial direction in the complex plane. Concentrating on the anti-diagonal elements ($n=-m-1$) just illustrates the point we want to make particularly well.

[12] See, e.g., P. Dennery and A. Krzywicki, *Mathematics for Physicists* (Dover, Mineola N.Y. 1996), ch.4 Sec.16.

[13] The frequency constraint in the Kronecker deltas in Eqs. (4.1) should actually read $\delta_{n_1,-n_2-1}$. Since the -1 is not important for any of our purposes, and disappears upon analytic continuation to real frequencies at zero temperature, we omit it here and throughout the remainder of Sec. IV.

[14] Note that this is not just a global gauge transformation, and therefore affects the physics.

[15] Some of the intermediate equations in Ref. 4 have a factor of i missing, but the final gap equation is correct.

[16] R. Heid, Z. Phys. **99**, 15 (1996).

[17] P.W. Anderson Phys. Rev. **112**, 1900 (1958); Y. Nambu, Phys. Rev. **117**, 648 (1960).

[18] This is useful because, for reasons that are not entirely obvious, our Gaussian approximation turns out to be conserving in the $r=3$ channel, but not in the $r=0$ one.

[19] Since there is some confusion concerning this point in the literature, it is worthwhile pointing out that the Anderson-Bogolubov mode, as the present derivation makes obvious,

is *not* the Goldstone mode, the latter being an unobservable mode in the particle-particle channel. Rather, it is a particle-hole channel manifestation of the Goldstone mode that comes about due to a coupling between the two channels.

[20] P. Coleman, E. Miranda, and A. Tsvelik, Phys. Rev. B **49**, 8955 (1994).

[21] This statement is true for our model which considers a local field $Q(\mathbf{x})$ only. If one allowed for nonlocal fields, then similar problems might arise in superconducting states with different order parameters.

[22] S. Weinberg, *The Quantum Theory of Fields*, vol.2 (Cambridge University Press Cambridge 1995), ch. 21.6.

[23] For a review, see, D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994), Sec. X.B.5.

[24] For a review, see, e.g., P.A. Lee and T.V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[25] B.L. Altshuler, A.G. Aronov, D.E. Khmelnitskii, and A.I. Larkin, in *Quantum Theory of Solids*, I.M. Lifshits (ed.) (MIR, Moscow 1982), p. 130.

[26] G. Bergmann, Phys. Rep. **101**, 1 (1984).